

# Finite Strings From Non-Chiral Mumford Forms

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## Abstract

We show that there is an infinite class of partition functions with world-sheet metric, space-time coordinates and first order systems, that correspond to volume forms on the moduli space of Riemann surfaces and are free of singularities at the Deligne-Mumford boundary. An example is the partition function with  $4 = 2(c_2 + c_3 + c_4 - c_5)$  space-time coordinates, a  $b$ - $c$  system of weight 3, one of weight 4 and a  $\beta$ - $\gamma$  system of weight 5. Such partition functions are derived from the mapping of the Mumford forms to non-factorized scalar forms on  $\mathcal{M}_g$  introduced in arXiv:1209.6049.

# 1 Introduction

Let us start by considering the following suggestion. Instead of considering the problems of infinities on the moduli space of curves  $\mathcal{M}_g$  of a given string theory, we consider the opposite question. Essentially, we consider the problem of finding the partition functions with a world sheet metric,  $D$  scalars,  $b$ - $c$  and  $\beta$ - $\gamma$  systems corresponding to volume forms on the moduli space of Riemann surfaces which are free of singularities at the Deligne-Mumford boundary. Such a question has a positive answer in view of the results of [1], where it was shown how to absorb, in a modular invariant way, the point dependence due to the insertion of  $b$  zero modes. As a consequence of that, the resulting partition functions involving the path integration on the world-sheet metric are well-defined volume forms  $\mathcal{M}_g$ .

The finding in [1][2] are in some sense a consequence of the basic observation by Yu. I. Manin [3][4] that the bosonic Polyakov measure [5] is (essentially) the same of the modulo square of the Mumford form [6].

The formulation of string and superstring perturbation theories face the hard problem of considering their effective characterization as geometrical objects on  $\mathcal{M}_g$ . The algebraic-geometrical structure of such a space, and of its Deligne-Mumford compactification  $\overline{\mathcal{M}}_g$ , provides a powerful tool, still largely unexplored, to check the consistency of the theory at any genus. Several questions concerning the definition of the theory, such as unitarity and structure of the singularities, are endowed in the geometry of  $\overline{\mathcal{M}}_g$ . Generally it is not true the opposite, there are subtle questions concerning  $\overline{\mathcal{M}}_g$  which may not appear in the purely field theoretic formulation. In this respect, in recent years it has been clear that the original formulations of the known string theories may map to other formulations whose elementary fields are different. The main example is the Berkovits pure spinor formulation of superstring theory [7][8]. Changing the field content leaving invariant the theory is a question strictly related to the Mumford isomorphism

$$\lambda_n \cong \lambda_1^{\otimes c_n} , \quad (1.1)$$

where  $\lambda_n$  are the determinant line bundles and  $c_n = 6n^2 - 6n + 1$ . Physically  $\lambda_n$  is associated to the partition function of a  $b$ - $c$  system of weight  $n$ , whereas the scalar target coordinates correspond to  $\lambda_1$ , the Hodge bundle. For each  $n$  the Mumford form  $\mu_{g,n}$  is the unique, up to a constant, holomorphic section of  $\lambda_n \otimes \lambda_1^{-c_n}$  nowhere vanishing on  $\mathcal{M}_g$ . It is useful to note that this implies that its inverse is also holomorphic.

Let  $C$  be a compact Riemann surface of genus  $g \geq 2$ . A further development concerning the Mumford isomorphism is the following map

$$\psi : \text{Sym}^n H^0(K_C) \longrightarrow H^0(K_C^n) , \quad (1.2)$$

which is surjective for  $g = 2$  and for  $C$  non-hyperelliptic of genus  $g > 2$ . Roughly speaking this provides another relation between the two sides of the Mumford isomorphism (1.1). It can be seen as a way to use the stringent properties of the Mumford forms  $\mu_{g,n}$  in considering the problem of characterizing the Jacobian (1.2). This is the essence of recent developments in the characterization of Riemann surfaces in  $\mathbb{P}^{g-1}$  and the related Schottky problem [9][10][11][12]. It has been shown in [2][1] that the map (1.2), together with the Mumford isomorphism, is crucial in constructing the string measures.

Here we show that the partition functions introduced in [1] include a class which is free of the singularities that usually arise when one considers the pinching of handles of the Riemann surface.

## 2 Singularity-free at the Deligne-Mumford boundary

In [1] it has been considering string theories with some underlying hidden symmetry which are still to be discovered from first principles. The preliminary basic step has been to classify all forms on  $\mathcal{M}_g$  satisfying some natural properties.

1. Since each string theory involves the path integration over the world-sheet metric, it should be a modular invariant  $(3g - 3, 3g - 3)$  form, i.e. a volume form on  $\mathcal{M}_g$ .
2. Such forms should correspond to determinants of laplacians associated to the space-time coordinates to  $b$ - $c$  and/or  $\beta$ - $\gamma$  systems of any conformal weight.
3. The combination of such determinants should be Weyl invariant.

The solution is based on the Weyl and modular invariant properties of the Bergman reproducing kernel

$$B(z, \bar{w}) = \sum_1^g \omega_j(z) (\tau_2^{-1})_{jk} \bar{\omega}_k(w) , \quad (2.1)$$

where  $\omega_1, \dots, \omega_g$ , are the canonically normalized holomorphic abelian differentials,  $\tau_2 \equiv \text{Im} \tau$ , with  $\tau$  the Riemann period matrix.

Let us show that

$$\int_{\mathcal{M}_g} \mathcal{Z}[\mathcal{J}] = \int Dg DX D\Psi \exp(-S[X] - S[\Psi]) , \quad (2.2)$$

corresponds to a volume form on  $\mathcal{M}_g$ . Here  $S[X]$  denotes the Polyakov action in

$$D = 26 + 2 \sum_{k \in \mathcal{I}} n_k c_k ,$$

dimensions, where  $c_k = 6k^2 - 6k + 1$  is (minus)  $1/2$  the central charge of the non-chiral ( $b$ - $c$ )  $\beta$ - $\gamma$  system of weight  $k$ .  $\mathcal{I}$  is the set of conformal weights  $k \in \mathbb{Q}$ ,  $\mathcal{J}$  the set of  $n_k \in \mathbb{Z}/2$ .  $D\Psi$  denotes the product on  $k \in \mathcal{I}$  of  $|n_k|$  copies of the non-chiral measures, including the zero mode insertions, of weight  $k$   $b$ - $c$  systems for  $n_k > 0$ , or  $\beta$ - $\gamma$  systems for  $n_k < 0$ .  $S[\Psi]$  is the sum of the corresponding non-chiral  $b$ - $c$  and  $\beta$ - $\gamma$  actions.

The string partition functions  $\mathcal{Z}[\mathcal{J}]$  correspond to the Polyakov partition function  $\mathcal{Z}_{Pol}$  times a rational function of determinants of laplacians. Namely

$$\mathcal{Z}[\mathcal{J}] = \mathcal{Z}_{Pol} \prod_{k \in \mathcal{I}} \mathcal{Z}_k^{n_k} . \quad (2.3)$$

In particular, it turns out that [1]

$$\mathcal{Z}_n = \int DX D b D \bar{b} D c D \bar{c} (b \bar{b})_n \exp(-S[X] - \frac{1}{2\pi} \int_C \sqrt{g} b \nabla_{1-n}^z c + c.c.) , \quad (2.4)$$

with

$$(b\bar{b})_n = \int_{C^{N_n}} \prod_j B^{1-n}(z_j, \bar{z}_j) b(z_j) \bar{b}(z_j) ,$$

coincides with

$$\mathcal{Z}_n = \left( \frac{\det' \Delta_0}{\mathcal{N}_0} \right)^{-c_n} \frac{\det \mathcal{M}_n \det' \Delta_{1-n}}{\det \mathcal{N}_n} , \quad (2.5)$$

where

$$(\mathcal{M}_n)_{jk} = \int_C \bar{\phi}_j^n(z) B^{1-n}(z, \bar{z}) \phi_k^n(z) ,$$

$$(\mathcal{N}_n)_{jk} = \int_C \bar{\phi}_j^n \rho^{1-n} \phi_k^n ,$$

with  $\phi_j^n$  are the zero modes associated to the  $b$ -field and  $\rho = 2g_{z\bar{z}}$  is the metric tensor in local complex coordinates, that is  $ds^2 = 2g_{z\bar{z}} dz d\bar{z}$ . It follows by (2.3) that the string partition functions (2.2) correspond to integral of Weyl invariant volume forms on  $\mathcal{M}_g$  [1]. This provides a manifestly intrinsic way to absorb the point dependence, that is the expression

$$B = {}^t \phi^1 \cdot Y^{-1} \cdot \bar{\phi}^1 ,$$

implies that  $(b\bar{b})_n$  depends only on the complex structure of the Riemann surface  $C$ .

Another result in [1] is that  $\mathcal{Z}[\mathcal{J}]$  can be expressed in terms of the building blocks of the Mumford forms  $F_{g,k}$ . In particular,

$$\mathcal{Z}[\mathcal{J}] = \frac{|F_{g,2}|^2}{(\det \tau_2)^{13}} \prod_{k \in \mathcal{I}} \left( \frac{|F_{g,k}|^2 \det \mathcal{M}_k}{(\det \tau_2)^{c_k}} \right)^{n_k} |\wedge^{3g-3} \phi_j^2|^2 , \quad (2.6)$$

which also provides the expression of  $\mathcal{Z}[\mathcal{J}]$  in terms of theta functions.

Let us shortly illustrate the Mumford forms. Let  $\phi_1^n, \dots, \phi_{N_n}^n$  be a basis of  $H^0(K_C^n)$ ,

$$N_n = (2n-1)(g-1) + \delta_{n1} .$$

Following Fay, we set [13]

$$\kappa[\phi^1] = \frac{\det \phi_i^1(z_j) \sigma(y) \prod_1^g E(y, z_i)}{\theta(\sum_1^g p_i - y - \Delta) \prod_1^g \sigma(z_i) \prod_{i < j}^g E(z_i, z_j)} ,$$

and, for  $n > 1$

$$\kappa[\phi^n] = \frac{\det \phi_i^n(z_j)}{\theta[\delta](\sum_1^{N_n} z_i - (2n-1)\Delta) \prod_1^{N_n} \sigma(p_i)^{2n-1} \prod_{i < j}^{N_n} E(z_i, z_j)} ,$$

where the above notation about theta functions, prime form and the  $g/2$ -differential  $\sigma$  is the standard one. Note that we added a dependence on the theta characteristic. Similar expressions hold for more general cases such as non-integer  $n$  [14]. Consider the universal curve  $\mathcal{C}_g$ , that is the  $(3g-2)$ -dimensional complex space built by placing over each point of  $\mathcal{M}_g$  the corresponding curve  $C$ . Let  $\pi$  be the map projecting  $\mathcal{C}_g$  to  $\mathcal{M}_g$ , and  $L_n = R\pi_*(K_{\mathcal{C}_g/\mathcal{M}_g}^n)$  the vector bundle on  $\mathcal{M}_g$  of rank  $N_n$  with fiber  $H^0(K_C^n)$  at the point

of  $\mathcal{M}_g$  representing  $C$ . Denote by  $\lambda_n = \det L_n$  the determinant line bundle. It turns out that, for each  $n$ , the form associated to the Mumford isomorphism is

$$\mu_{g,n} = F_{g,n}[\phi^n] \frac{\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n}{(\omega_1 \wedge \cdots \wedge \omega_g)^{c_n}} , \quad (2.7)$$

where

$$F_{g,n}[\phi^n] = \frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]} . \quad (2.8)$$

Such quantities naturally appear in describing the geometry of curves in  $\mathbb{P}^{g-1}$  and in characterizing the Jacobian locus [10][11][15][12]. These also appear in the string measures. In [3][4][16][14] it has been shown that

$$\left( \frac{\det' \Delta_0}{\mathcal{N}_0 \det \mathcal{N}_1} \right)^{-c_n} \frac{\det' \Delta_{1-n}}{\det \mathcal{N}_{1-n} \det \mathcal{N}_n} = \left| \frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]} \right|^2 . \quad (2.9)$$

Eq.(2.9) expresses determinants of laplacians in terms of theta functions whereas Eq.(2.7) establishes the relation between determinants of laplacians and the Mumford forms. Let us note that there is a lot of literature on string determinants, loop amplitudes and  $b$ - $c$  systems, see for example [19]-[33]. In the following we will consider  $n$  such that  $h^0(K_C^{1-n}) = 0$  and drop the term  $\det \mathcal{N}_{1-n}$  in (2.9).

The above prescription of absorbing the zero modes is essentially the only well-defined recipe on any Riemann surface. Nevertheless, there is a related approach which is defined on the compact Riemann surfaces of genus two and on the non-hyperelliptic compact Riemann surfaces with  $g > 2$ , these are called canonical curves. Instead of integrating with  $B^{1-n}(z_j, \bar{z}_j)$  each pair  $b(z_j)\bar{b}(z_j)$  of the zero mode insertions, one may divide them by the determinant of  $B^{(n)}(z_j, \bar{z}_k)$ , denoting the  $n$ -fold Hadamard product of  $B(z_j, \bar{z}_k)$ . In particular, it turns out that

$$V_n = \int DX D b D \bar{b} D c D \bar{c} \frac{\prod_i b(z_i) \bar{b}(z_i)}{\det B^{(n)}(z_j, \bar{z}_k)} \exp(-S[X] - \frac{1}{2\pi} \int_C \sqrt{g} b \nabla_{1-n}^z c + c.c.) , \quad (2.10)$$

where  $S[X]$  is the Polyakov action in  $2c_n$  dimensions, is a well-defined prescription on canonical curves. This implies that

$$\int_{\mathcal{M}_g} V[\mathcal{J}] = \int Dg DX D\Psi \exp(-S[X] - S[\Psi]) , \quad (2.11)$$

where now the Polyakov action is in  $D = 26 + 2 \sum_{k \in \mathcal{I}} n_k c_k$  dimensions, is a Weyl anomaly free partition function. We have

$$V[\mathcal{J}] = \mathcal{Z}_{Pol} \prod_{k \in \mathcal{I}} V_k^{n_k} , \quad (2.12)$$

which are volume forms on  $\mathcal{M}_g$ . In particular, [1]

$$V_n(\tau) = \frac{K_1^{(2n-1)^2}}{K_n} \frac{1}{(\det \tau_2)^{2n(n-1)}} , \quad (2.13)$$

which are a  $(0,0)$ -form on  $\mathcal{M}_g$ . Here

$$K_n = \frac{\det B^{\text{on}}(z_i, \bar{z}_j)}{|\det \phi_j^n(z_k)|^2} |\kappa[\phi^n]|^2, \quad (2.14)$$

where the determinant of  $B^{\text{on}}(z_j, \bar{z}_k)$  is taken with the indices  $j, k$  ranging from 1 to  $N_n$ . Note that

$$K_1 = \frac{|\kappa[\omega]|^2}{\det \tau_2}.$$

One may check that  $V_n(\tau)$  is obtained from the modulo square of the Mumford forms by a non-chiral mapping [1].

We now look at the  $\mathcal{Z}[\mathcal{J}]$  which are free of singularities at the Deligne-Mumford boundary. Following Fay [14], we consider their behavior at  $\partial\overline{\mathcal{M}}_g$ . Fay proved, using Bers like basis  $\phi_t^n = \{\phi_{i,t}^n\}_{i \in I_{N_n}}$  for  $H^0(K_C^n)$ , that  $F_{g,n}$  has a pole of order  $n(n-1)/2$  in the plumbing fixture parameter  $t$ .

Let us consider the degeneration limits. First, by approaching the generic point on a irreducible singular curve obtained by identifying two points  $a, b$  on a smooth genus  $g-1$  curve. In this case

$$F_{g,n}[\phi_t^n, \omega] \sim t^{-n(n-1)/2} \frac{E(a, b)^{n-n^2}}{(2\pi i)^{(2n-1)^2}} F_{g-1,n}[\phi^n, \omega]. \quad (2.15)$$

In the case of degeneration corresponding to a reducible singular curve obtained by identifying points on two smooth curves of genus  $g_1$  and  $g-g_1$ , we have

$$F_{g,n}[\phi_t^n, \omega] \sim \epsilon t^{-n(n-1)/2} F_{g-g_1,n}[\phi^n \omega] G_{g_1,n}[\phi^n, \omega], \quad (2.16)$$

where  $\epsilon$  is a fixed  $(2g-2)$ th root of unity. We now consider rational functions of Mumford forms

$$\mathcal{F}_g[\phi^k, \omega, \mathcal{J}] = \frac{\prod_{k \in \mathcal{I}} [F_{g,k}^{n_k}[\phi^k, \omega] \wedge^{\text{max}} \phi_j^k]}{(\omega_1 \wedge \dots \wedge \omega_g)^{d/2}}, \quad (2.17)$$

$n_k \in \mathbb{Z}$ . Note that

$$d = 2 \sum_{k \in \mathcal{I}} n_k c_k. \quad (2.18)$$

Since the Mumford forms on  $\mathcal{M}_g$  are holomorphic and non-vanishing, imposing the absence of singularities for  $\mathcal{F}_g$  on  $\overline{\mathcal{M}}_g$  is equivalent to require holomorphicity at the Deligne-Mumford boundary. This gives

$$\sum_{k \in \mathcal{I}} n_k k(k-1) = \frac{1}{6} \sum_{k \in \mathcal{I}} n_k (c_k - 1) \leq 0. \quad (2.19)$$

Eq.(2.18) and (2.19) imply

$$d \leq 2 \sum_{k \in \mathcal{I}} n_k. \quad (2.20)$$

We have

1. If  $\sum_{k \in \mathcal{I}} n_k k(k-1) = 0$ , then

$$\mathcal{F}_g[\phi^k, \omega, \mathcal{J}] = \mathcal{F}_g[\phi^k, \phi^1, \mathcal{J}] , \quad (2.21)$$

with  $\phi_1^1, \dots, \phi_g^1$  an arbitrary basis of  $H^0(K_C)$ .

2.  $\sum_{k \in \mathcal{I}} n_k k(k-1) = 0$  then at the non-separating node of the Deligne-Mumford boundary

$$\mathcal{F}_g \sim \frac{1}{(2\pi i)^{d/2}} \mathcal{F}_{g-1} , \quad (2.22)$$

whereas at the separating node

$$\mathcal{F}_g \sim \epsilon \mathcal{F}_{g-g_1} \mathcal{F}_{g_1} . \quad (2.23)$$

The invariance (2.21) follows by observing that if (2.19) is satisfied, then, due to the term  $\det \phi_j^1(z_k)$  at the numerator of  $\kappa[\phi^1]$

$$\mathcal{F}_g = \prod_{k \in \mathcal{I}} \left( \frac{\wedge^{max} \phi_j^k}{\kappa[\phi^k]} \right)^{n_k} ,$$

where  $n_1 \equiv -d/2$ . The factorization properties (2.22) and (2.23) follow by (2.15) and (2.16).

The previous analysis implies that the partition functions

$$\mathcal{Z}[\mathcal{J}] ,$$

with

$$\sum_{k \in \mathcal{I}} n_k (c_k - 1) + 12 \leq 0 , \quad (2.24)$$

are finite at the Deligne-Mumford boundary. In this case

$$D \leq 2 + 2 \sum_{k \in \mathcal{I}} n_k .$$

The first partition function with  $D = 4$  space-time coordinates which is finite at  $\partial \overline{\mathcal{M}}_g$  is the one with  $n_2 = 0$ ,  $n_3 = n_4 = 1$  and  $n_5 = -1$ , that is

$$D = 2(c_2 + c_3 + c_4 - c_5) = 4 ,$$

corresponding to

$$\int_{\mathcal{M}_g} \mathcal{Z}[1_3, 1_4, -1_5] = \int Dg DX D\Psi \exp(-S[X] - S[\Psi]) . \quad (2.25)$$

$S[X]$  is the four dimensional Polyakov action, whereas  $S[\Psi]$  is the sum of the action of a non-chiral  $b$ - $c$  system of weight 3, another of weight 4 and a  $\beta$ - $\gamma$  system of weight 5. Note that in this case

$$\sum_{k \in \mathcal{I}} n_k (c_k - 1) + 12 = 0 ,$$

so that  $\mathcal{Z}_g[1_3, 1_4, -1_5]$  satisfies the factorization properties like the ones in (2.22) and (2.23).

A similar analysis extends to  $V[\mathcal{J}]$ . However, since  $\det B^{on}(z_j, \bar{z}_k)$  vanishes on the hyperelliptic locus with  $g \neq 2$ , the above results can be extended to  $V[\mathcal{J}]$  in the case of canonical curves. The extension to the full  $\mathcal{M}_g$  requires some further condition when considering the hyperelliptic loci of  $\mathcal{M}_g$ . Let us stress that  $V[-1_2]$ , that in [1] has been shown to coincide with the volume form on  $\mathcal{M}_g$  induced by the Siegel metric, corresponds to a finite string in  $D = 0$ . It follows that

$$Vol_S(\mathcal{M}_g) = \int_{\mathcal{M}_g} \mathcal{Z}[-1_2] , \quad (2.26)$$

is the Siegel volume of  $\mathcal{M}_g$ . This may indicate that, as discussed in [2] in the case of the Polyakov and supersymmetric strings, even the above models can be obtained, by a reduction mechanism, from volume forms on the Siegel upper half-space.

The partition functions  $\mathcal{Z}[\mathcal{J}]$  and  $V[\mathcal{J}]$  represent an infinite class of volume forms on  $\mathcal{M}_g$ . Studying their symmetries and spectra may also lead to understand some of the more recent questions of superstring perturbation theory [34]-[62].

## Acknowledgements

I thank Giulio Bonelli, Giulio Codogni, Maurizio Cornalba, Paolo di Vecchia, Pietro Grassi, Samuel Grushevsky, Ian Morrison, Paolo Pasti, Riccardo Salvati Manni, Augusto Sagnotti, Dima Sorokin, Mario Tonin and Roberto Volpato for helpful comments and discussions. This research is supported by the Padova University Project CPDA119349 and by the MIUR-PRIN contract 2009-KHZKRX.



# References

- [1] M. Matone, Modular Invariant Regularization of String Determinants and the Serre GAGA Principle, arXiv:1209.6049 [hep-th].
- [2] M. Matone, Extending the Belavin-Knizhnik “wonderful formula” by the characterization of the Jacobian, arXiv:1208.5994 [hep-th], to appear in JHEP.
- [3] Y. I. Manin, The partition function of the Polyakov string can be expressed in terms of theta functions, Phys. Lett. B **172** (1986) 184.
- [4] A. A. Beilinson and Y. I. Manin, The Mumford form and the Polyakov measure in string theory, Commun. Math. Phys. **107** (1986) 359.
- [5] A. M. Polyakov, Quantum geometry of bosonic strings, Phys. Lett. B **103** (1981) 207.
- [6] D. Mumford, Stability of projective varieties, Enseign. Math. **23** (1977) 39.
- [7] N. Berkovits, Super-Poincare covariant quantization of the superstring, JHEP **0004** (2000) 018.
- [8] M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings, Nucl. Phys. B **639** (2002) 182.
- [9] M. Matone and R. Volpato, Linear relations among holomorphic quadratic differentials and induced Siegel’s metric on  $\mathcal{M}_g$ , J. Math. Phys. **52** (2011) 102305.
- [10] M. Matone and R. Volpato, Determinantal characterization of canonical curves and combinatorial theta identities, Math. Ann. DOI: 10.1007/s00208-012-0787-z.
- [11] M. Matone and R. Volpato, The singular locus of the theta divisor and quadrics through a canonical curve, arXiv:0710.2124 [math.AG].
- [12] M. Matone and R. Volpato, Vector-valued modular forms from the Mumford form, Schottky-Igusa form, product of thetanullwerte and the amazing Klein formula, arXiv:1102.0006 [math.AG], to appear in the Proc. of the AMS.
- [13] J. Fay, Theta Functions on Riemann surfaces, Springer Lecture Notes in Math. **352** 1973.
- [14] J. Fay, Kernel functions, analytic torsion and moduli spaces, Mem. Am. Math. Soc. **96** (1992).
- [15] N. I. Shepherd-Barron, Thomae’s formulae for non-hyperelliptic curves and spinorial square roots of theta-constants on the moduli space of curves, arXiv:0802.3014 [math.AG].
- [16] A. A. Belavin and V. G. Knizhnik, Algebraic geometry and the geometry of quantum strings, Phys. Lett. B **168** (1986) 201.

- [17] V. G. Knizhnik, Multiloop amplitudes in the theory of quantum strings and complex geometry, *Sov. Phys. Usp.* **32** (1989) 945 [*Usp. Fiz. Nauk* **159** (1989) 401].
- [18] E. P. Verlinde and H. L. Verlinde, Chiral Bosonization, Determinants and the String Partition Function, *Nucl. Phys. B* **288** (1987) 357.
- [19] E. D'Hoker and D. H. Phong, On Determinants Of Laplacians On Riemann Surfaces, *Commun. Math. Phys.* **104** (1986) 537; Multiloop Amplitudes for the Bosonic Polyakov String, *Nucl. Phys. B* **269** (1986) 205.
- [20] E. D'Hoker and D. H. Phong, The Geometry of String Perturbation Theory, *Rev. Mod. Phys.* **60** (1988) 917.
- [21] L. Alvarez-Gaume, G. W. Moore and C. Vafa, Theta Functions, Modular Invariance and Strings, *Commun. Math. Phys.* **106** (1986) 1.
- [22] L. Alvarez-Gaumé, J. B. Bost, G. W. Moore, P. C. Nelson and C. Vafa, Bosonization In Arbitrary Genus, *Phys. Lett. B* **178** (1986) 41; Bosonization on Higher Genus Riemann Surfaces, *Commun. Math. Phys.* **112** (1987) 503.
- [23] A. Morozov, Explicit formulae for one, two, three and four loop string amplitudes, *Phys. Lett. B* **184** (1987) 171.
- [24] P. Di Vecchia, M. Frau, A. Lerda and S. Sciuto, A Simple Expression For The Multiloop Amplitude In The Bosonic String, *Phys. Lett. B* **199** (1987) 49.
- [25] P. Di Vecchia, M. Frau, A. Lerda and S. Sciuto, N String Vertex And Loop Calculation In The Bosonic String, *Nucl. Phys. B* **298** (1988) 527.
- [26] P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda and S. Sciuto, N String, G Loop Vertex For The Fermionic String, *Phys. Lett. B* **211** (1988) 301.
- [27] K. Roland, Two And Three Loop Amplitudes In Covariant Loop Calculus, *Nucl. Phys. B* **313** (1989) 432.
- [28] P. Di Vecchia, L. Magnea, A. Lerda, R. Russo and R. Marotta, String techniques for the calculation of renormalization constants in field theory, *Nucl. Phys. B* **469** (1996) 235 [[hep-th/9601143](#)].
- [29] L. Alvarez-Gaume, C. Gomez, G. W. Moore and C. Vafa, *Nucl. Phys. B* **303** (1988) 455.
- [30] L. Alvarez-Gaumé, C. Gomez, P. C. Nelson, G. Sierra and C. Vafa, Fermionic Strings In The Operator Formalism, *Nucl. Phys. B* **311** (1988) 333.
- [31] L. Bonora, A. Lugo, M. Matone and J. Russo, A Global Operator Formalism on Higher Genus Riemann Surfaces: b-c Systems, *Commun. Math. Phys.* **123** (1989) 329.
- [32] L. Bonora, M. Matone, F. Toppan and K. Wu, B-c System Approach To Minimal Models. 1. The Genus Zero Case, *Phys. Lett. B* **224** (1989) 115;

- [33] L. Bonora, M. Matone, F. Toppan and K. Wu, Real weight  $b$ - $c$  systems and Conformal Field Theory in higher genus, Nucl. Phys. B **334** (1990) 717.
- [34] E. D'Hoker and D. H. Phong, Two-loop superstrings I, main formulas, Phys. Lett. B **529** (2002) 241; Two-loop superstrings II, the chiral measure on moduli space, Nucl. Phys. B **636** (2002) 3; Two-loop superstrings III, slice independence and absence of ambiguities, Nucl. Phys. B **636** (2002) 61; Two-loop superstrings IV, the cosmological constant and modular forms, Nucl. Phys. B **639** (2002) 129; Two-loop superstrings V: gauge slice independence of the N-point function, Nucl. Phys. B **715** (2005) 91.
- [35] E. D'Hoker and D. H. Phong, Aszygies, modular forms, and the superstring measure. I, Nucl. Phys. B **710** (2005) 58.
- [36] E. D'Hoker and D. H. Phong, Aszygies, modular forms, and the superstring measure. II, Nucl. Phys. B **710** (2005) 83.
- [37] E. D'Hoker and D. H. Phong, Two-loop superstrings VI: non-renormalization theorems and the 4-point function, Nucl. Phys. B **715** (2005) 3.
- [38] E. D'Hoker and D. H. Phong, Complex geometry and supergeometry, Current Developments in Math., **2005** (2007) 1.
- [39] E. D'Hoker and D. H. Phong, Two-loop superstrings VII, cohomology of chiral amplitudes, Nucl. Phys. B **804** (2008) 421.
- [40] E. D'Hoker, M. Gutperle and D. H. Phong, Two-loop superstrings and S-duality, Nucl. Phys. B **722** (2005) 81.
- [41] M. Matone and R. Volpato, Higher genus superstring amplitudes from the geometry of moduli spaces, Nucl. Phys. B **732** (2006) 321.
- [42] S. L. Cacciatori and F. Dalla Piazza, Two loop superstring amplitudes and  $S(6)$  representations, Lett. Math. Phys. **83** (2008) 127.
- [43] S. L. Cacciatori, F. Dalla Piazza and B. van Geemen, Modular forms and three loop superstring amplitudes, Nucl. Phys. B **800** (2008) 565; Genus four superstring measures, Lett. Math. Phys. **85** (2008) 185.
- [44] S. Grushevsky, Superstring scattering amplitudes in higher genus, Commun. Math. Phys. **287** (2009) 749.
- [45] R. Salvati Manni, Remarks on superstring amplitudes in higher genus, Nucl. Phys. B **801** (2008) 163.
- [46] A. Morozov, NSR Superstring Measures Revisited, JHEP **0805** (2008) 086.
- [47] F. Dalla Piazza and B. van Geemen, Siegel modular forms and finite symplectic groups, Adv. Theor. Math. Phys. **13** (2009) 1771.

- [48] A. Morozov, NSR measures on hyperelliptic locus and non-renormalization of 1,2,3-point functions, *Phys. Lett. B* **664** (2008) 116.
- [49] F. D. Piazza, More on superstring chiral measures, *Nucl. Phys. B* **844** (2011) 471.
- [50] F. D. Piazza, D. Girola and S. L. Cacciatori, Classical theta constants vs. lattice theta series, and super string partition functions, *JHEP* **1011** (2010) 082.
- [51] M. Oura, C. Poor, R. Salvati Manni and D. Yuen, Modular forms of weight 8 for  $\Gamma_g(1, 2)$ , *Math. Ann.* **346** (2010) 477.
- [52] P. Dunin-Barkowski, A. Morozov and A. Sleptsov, Lattice theta constants vs Riemann theta constants and NSR superstring measures, *JHEP* **0910** (2009) 072.
- [53] S. Grushevsky and R. Salvati Manni, The vanishing of two-point functions for three-loop superstring scattering amplitudes, *Commun. Math. Phys.* **294** (2010) 393.
- [54] M. Matone and R. Volpato, Superstring measure and non-renormalization of the three-point amplitude, *Nucl. Phys. B* **806** (2009) 735.
- [55] S. Grushevsky and R. S. Manni, The superstring cosmological constant and the Schottky form in genus 5, *Amer. J. Math.* **133** (2011) 1007.
- [56] C. Poor and D. S. Yuen, Binary forms and the hyperelliptic superstring ansatz, arXiv:0911.4545 [math.NT].
- [57] M. Matone and R. Volpato, Getting superstring amplitudes by degenerating Riemann surfaces, *Nucl. Phys. B* **839** (2010) 21.
- [58] G. Codogni and N. I. Shepherd-Barron, The absence of stable Schottky forms, arXiv:1112.6137 [math.AG].
- [59] P. Dunin-Barkowski, A. Sleptsov and A. Stern, NSR superstring measures in genus 5, arXiv:1208.2324 [hep-th].
- [60] E. Witten, Notes On Supermanifolds and Integration, arXiv:1209.2199 [hep-th].
- [61] E. Witten, Notes On Super Riemann Surfaces And Their Moduli, arXiv:1209.2459 [hep-th].
- [62] E. Witten, Superstring Perturbation Theory Revisited, arXiv:1209.5461 [hep-th].